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Application of Homotopy Perturbation Sumudu Transform Method for Solving Heat and Wave-Like Equations

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ABSTRACT

In this paper, we use the homotopy perturbation sumudu transform method (HPSTM) to solve heat and wave-like equations. The proposed scheme finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. Several examples are given to verify the reliability and efficiency of the method. The fact that the proposed technique solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method.

Keywords: homotopy perturbation sumudu transform method, sumudu transform, heat and wave-like equations, He's Polynomials.

1. INTRODUCTION

The heat and wave-like models are the integral part of applied sciences and arise in various physical phenomena. Several techniques including spectral, characteristic, modified variational iteration, Adomian's decomposition method and He's polynomials have been used for solving these problems(see Noor and Mohyud-Din (2008), Wazwaz and Gorguis (2004), Wilcox (1970) and Mohyud-Din (2009)) and references therein. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results and huge computational work. He (1999, 2003, 2004) and references therein developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems.

It is worth mentioning that the HPM is applied without any discretization, restrictive assumption or transformation and is free from round off errors. The homotopy perturbation method is also combined with the well-known Laplace transformation method (Madani and Fathizadeh (2010) and Khan and Wu (2011)) to produce a highly effective technique for handling many nonlinear problems. Very recently Singh, Kumar and Sushila (2011) have introduced, a new technique called homotopy perturbation sumudu transform method (HPSTM) for solving nonlinear equations. It is worth mentioning that HPSTM is an elegant combination of the sumudu transform method, the homotopy perturbation method and He's polynomials and is mainly due to Ghorbani and Saberi-Nadjafi (2007) and Ghorbani (2009).

The use of He's polynomials in the nonlinear term was first introduced by Ghorbani and Saberi-Nadjafi (2007) and Ghorbani (2009). HPSTM provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact and approximate solutions for nonlinear equations. Inspired and motivated by the ongoing research in this area, we apply HPSTM for solving the heat and wave-like equations in the present article. Several examples are given to verify the reliability and efficiency of the technique.

2. SUMUDU TRANSFORM

In early 90's, Watugala (1998) introduced a new integral transform, named the sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The sumudu transform is defined over the set of functions

$$A = \{f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j}, \text{if } t \in (-1)^j \times [0,\infty)\}$$

by the following formula

$$\bar{f}(u) = S[f(t)] = \int_0^\infty f(ut)\, e^{-t}\, dt, \; u \in (-\tau_1, \tau_2). \tag{1}$$

Some of the properties were established by Weerakoon in Kilicman *et al.* (2011) and Weerakoon (1994). In Asiru (2004), further fundamental properties of this transform were also established. Similarly, this transform was applied to the one-dimensional neutron transport equation in Kadem

(2005). In fact it was shown that there is strong relationship between Sumudu and other integral transform (see Kilicman and Eltayeb (2010). In particular the relation between Sumudu transform and Laplace transforms was proved in Kilicman and Eltayeb (2010).

Further, in Eltayeb *et al.* (2010), the Sumudu transform was extended to the distributions and some of their properties were also studied in Kilicman *et al.* (2010). Recently, this transform is applied to solve the system of differential equations (see Kilicman *et al.* (2010)).

Note that a very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except the factor n (see Zhang (2007)).

Thus if $f(t)=\sum_{n=0}^{\infty} a_n t^n$ then $F(u)=\sum_{n=0}^{\infty} n!a_n u^n$, see Kilicman and Eltayeb (2010). Similarly, the Sumudu transform sends combinations, $C(m, n)$, into permutations, $P(m,n)$ and hence it will be useful in the discrete systems.

3. HOMOTOPY PERTURBATION SUMUDU TRANSFORM METHOD (HPSTM)

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of the form

$$DU(x,t)+RU(x,t)+NU(x,t)=g(x,t), \tag{2}$$

$$U(x,0)=h(x), \quad U_t(x,0)=f(x),$$

where D is the second order linear differential operator $D=\partial^2/\partial t^2$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking the sumudu transform on both sides of Equation (2), we get

$$S[DU(x,t)]+S[RU(x,t)]+S[NU(x,t)]=S[g(x,t)]. \tag{3}$$

Using the differentiation property of the sumudu transform and above initial conditions, we have

$$S[U(x,t)]=u^2 S[g(x,t)]+h(x)+uf(x)-u^2 S\big[[RU(x,t)]+[NU(x,t)]\big]. \tag{4}$$

Now, applying the inverse sumudu transform on both sides of Equation (4), we get

$$U(x,t)=G(x,t)-S^{-1}\left[u^2\, S[RU(x,t)+NU(x,t)]\right], \tag{5}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method

$$U(x,t)=\sum_{n=0}^{\infty} p^n U_n(x,t) \tag{6}$$

and the nonlinear term can be decomposed as

$$NU(x,t)=\sum_{n=0}^{\infty} p^n H_n(U), \tag{7}$$

for some He's polynomials $H_n(U)$ (see Ghorbani (2009) and Mohyud-Din *et al.* (2009)) that are given by

$$H_n(U_0,U_1,\ldots,U_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i U_i\right)\right]_{p=0},\ \ n=0,1,2,3,\ldots \tag{8}$$

Substituting Equations (6) and (7) in Equation (5), we get

$$\sum_{n=0}^{\infty} p^n U_n(x,t) \\ =G(x,t)-p\left(S^{-1}\left[u^2\, S\left[R\sum_{n=0}^{\infty} p^n U_n(x,t)+\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right), \tag{9}$$

which is the coupling of the sumudu transform and the homotopy perturbation method using He's polynomials.

By comparing the coefficient of like powers of p, the following approximations are obtained

$$\begin{aligned}
&p^0 : U_0(x,t) = G(x,t),\\
&p^1 : U_1(x,t) = -S^{-1}\left[u^2 S[RU_0(x,t) + H_0(U)]\right],\\
&p^2 : U_2(x,t) = -S^{-1}\left[u^2 S[RU_1(x,t) + H_1(U)]\right],\\
&p^3 : U_3(x,t) = -S^{-1}\left[u^2 S[RU_2(x,t) + H_2(U)]\right],\\
&\vdots
\end{aligned} \tag{10}$$

4. NUMERICAL APPLICATIONS

In this section, we apply the homotopy perturbation Sumudu transform method [HPSTM] for solving heat and wave-like equations.

Example 4.1. Consider the following one-dimensional initial boundary value problem which describes the heat-like models (Noor and Mohyud-Din (2008) and Wazwaz and Gorguis (2004).

$$U_t = \frac{1}{2}x^2 U_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{11}$$

with boundary conditions

$$U(0,t) = 0, \quad U(1,t) = e^t, \tag{12}$$

and the initial condition

$$U(x,0) = x^2. \tag{13}$$

Taking the sumudu transform on both sides of equation (11) subject to the initial condition, we have

$$S[U(x,t)] = x^2 + \frac{1}{2}x^2 u\, S[U_{xx}]. \tag{14}$$

The inverse of sumudu transform implies that

$$U(x,t) = x^2 + \frac{1}{2}x^2 S^{-1}\left[u\,S[U_{xx}]\right]. \tag{15}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x^2 + p\left(\frac{1}{2}x^2 S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}\right]\right]\right). \tag{16}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} p^0 &: U_0(x,t) = x^2, \\ p^1 &: U_1(x,t) = \frac{1}{2}x^2 S^{-1}\left[u\,S[(U_0)_{xx}]\right] = x^2\,t, \\ p^2 &: U_2(x,t) = \frac{1}{2}x^2 S^{-1}\left[u\,S[(U_1)_{xx}]\right] = x^2\,\frac{t^2}{2!} \end{aligned} \tag{17}$$

Proceeding in a similar manner, we have

$$\begin{aligned} p^3 &: U_3(x,t) = x^2\,\frac{t^3}{3!}, \\ p^4 &: U_4(x,t) = x^2\,\frac{t^4}{4!}, \\ &\vdots \end{aligned} \tag{18}$$

Therefore the solution $U(x,t)$ is given by

$$U(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right), \tag{19}$$

in a series form, and

$$U(x,t) = x^2 e^t, \tag{20}$$

in closed form.

Example 4.2. Consider the following two-dimensional initial boundary value problem which describes the heat-like models (Noor and Mohyud-Din (2008) and Wazwaz and Gorguis (2004)).

$$U_t = \frac{1}{2}\left(y^2 U_{xx} + x^2 U_{yy}\right), \qquad 0 < x,\ y < 1,\ \ t > 0, \tag{21}$$

with boundary conditions

$$\begin{aligned} U_x(0,y,t) &= 0, \quad & U_x(1,y,t) &= 2\sinh t, \\ U_y(x,0,t) &= 0, \quad & U_y(x,1,t) &= 2\cosh t, \end{aligned} \tag{22}$$

and initial condition

$$U(x,y,0) = y^2. \tag{23}$$

In a similar way as above, we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,y,t) = y^2 + p\Bigg(&\frac{1}{2} y^2 S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{xx}\right]\right] \\ &+\frac{1}{2} x^2 S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{yy}\right]\right]\Bigg). \end{aligned} \tag{24}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} p^0 &: U_0(x,y,t) = y^2, \\ p^1 &: U_1(x,y,t) = x^2 t, \\ p^2 &: U_2(x,y,t) = y^2 \frac{t^2}{2!}, \\ p^3 &: U_3(x,y,t) = x^2 \frac{t^3}{3!}, \\ p^4 &: U_4(x,y,t) = y^2 \frac{t^4}{4!}, \\ &\vdots \end{aligned} \tag{25}$$

Therefore the solution $U(x,y,t)$ is given by

$$U(x,y,t) = x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) + y^2\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right), \tag{26}$$

which is in series form, and

$$U(x,y,t) = x^2 \sinh t + y^2 \cosh t, \tag{27}$$

in closed form.

Example 4.3. Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the heat-like models (Noor and Mohyud-Din (2008) and Wazwaz and Gorguis (2004)).

$$U_t = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right), \ 0 < x,\ y,\ z < 1,\ \ t > 0, \tag{28}$$

subject to the following boundary conditions

$$\begin{aligned} &U(0,y,z,t) = 0, \quad U(1,y,z,t) = y^4 z^4 (e^t - 1), \\ &U(x,0,z,t) = 0, \quad U(x,1,z,t) = x^4 z^4 (e^t - 1), \\ &U(x,y,0,t) = 0, \quad U(x,y,1,t) = x^4 y^4 (e^t - 1), \end{aligned} \tag{29}$$

and the initial condition

$$U(x,y,z,0) = 0. \tag{30}$$

In a similar way as above, we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,y,z,t) = x^4 y^4 z^4 t + p\Bigg(&\frac{1}{36} x^2 S^{-1}\left[u\, S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right)_{xx} \right]\right] \\ &+ \frac{1}{36} y^2 S^{-1}\left[u\, S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right)_{yy} \right]\right] \\ &+ \frac{1}{36} z^2 S^{-1}\left[u\, S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right)_{zz} \right]\right]\Bigg). \end{aligned} \tag{31}$$

Comparing the coefficients of like powers of p, we have

$$p^0 : U_0(x,y,z,t) = x^4 y^4 z^4 t,$$

$$p^1 : U_1(x,y,z,t) = x^4 y^4 z^4 \frac{t^2}{2!},$$

$$p^2 : U_2(x, y, z, t) = x^4 y^4 z^4 \frac{t^3}{3!}, \tag{32}$$
$$p^3 : U_3(x, y, z, t) = x^4 y^4 z^4 \frac{t^4}{4!},$$
$$p^4 : U_4(x, y, z, t) = x^4 y^4 z^4 \frac{t^5}{5!},$$
$$\vdots$$

Therefore the solution $U(x, y, z, t)$ is given by

$$U(x, y, z, t) = x^4 y^4 z^4 \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots \right), \tag{33}$$

in a series form, and

$$U(x, y, z, t) = x^4 y^4 z^4 (e^t - 1), \tag{34}$$

in closed form.

Example 4.4. Consider the following one-dimensional initial boundary value problem which describes the wave-like models (Noor and Mohyud-Din (2008) and Wazwaz and Gorguis (2004))

$$U_{tt} = \frac{1}{2} x^2 U_{xx}, \quad 0 < x < 1, \ t > 0, \tag{35}$$

subject to the boundary conditions

$$U(0,t) = 0, \quad U(1,t) = 1 + \sinh t, \tag{36}$$

and the initial conditions

$$U(x,0) = x, \qquad U_t(x,0) = x^2. \tag{37}$$

In a similar way as above, we have

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x + x^2 t + p\left(\frac{1}{2} x^2 S^{-1} \left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t) \right)_{xx} \right] \right] \right). \tag{38}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} p^0 &: U_0(x,t) = x + x^2 t, \\ p^1 &: U_1(x,t) = x^2 \frac{t^3}{3!}, \\ p^2 &: U_2(x,t) = x^2 \frac{t^5}{5!}, \\ p^3 &: U_3(x,t) = x^2 \frac{t^7}{7!}, \\ &\vdots \end{aligned} \tag{39}$$

Therefore the solution $U(x,t)$ is given by

$$U(x,t) = x + x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots \right), \tag{40}$$

in series form, and

$$U(x,t) = x + x^2 \sinh t, \tag{41}$$

in closed form.

Example 4.5. Consider the following two-dimensional initial boundary value problem which describes the wave-like models (Noor and Mohyud-Din (2008) and Wazwaz and Gorguis (2004)

$$U_{tt} = \frac{1}{12}\left(x^2 U_{xx} + y^2 U_{yy} \right), \qquad 0 < x,\ y < 1,\ \ t > 0, \tag{42}$$

subject to the Neumann boundary conditions

$$\begin{aligned} U_x(0,y,t) &= 0, \quad & U_x(1,y,t) &= 4\cosh t, \\ U_y(x,0,t) &= 0, \quad & U_y(x,1,t) &= 4\sinh t, \end{aligned} \tag{43}$$

and the initial conditions

$$U(x,y,0) = x^4, \qquad U_t(x,y,0) = y^4. \tag{44}$$

In a similar way as above, we have

$$\sum_{n=0}^{\infty} p^n U_n(x,y,t) = x^4 + y^4 t + p\left(\frac{1}{12}x^2 S^{-1}\left[u^2\, S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{xx}\right]\right]\right.$$
$$\left. +\frac{1}{12}y^2 S^{-1}\left[u^2\, S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{yy}\right]\right]\right). \tag{45}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned}
p^0 &: U_0(x,y,t) = x^4 + y^4 t,\\
p^1 &: U_1(x,y,t) = x^4\frac{t^2}{2!} + y^4\frac{t^3}{3!},\\
p^2 &: U_2(x,y,t) = x^4\frac{t^4}{4!} + y^4\frac{t^5}{5!},\\
p^3 &: U_3(x,y,t) = x^4\frac{t^6}{6!} + y^4\frac{t^7}{7!},\\
&\vdots
\end{aligned} \tag{46}$$

Therefore the solution $U(x,y,t)$ is given by

$$U(x,y,t) = x^4\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right) + y^4\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots\right), \tag{47}$$

in series form, and

$$U(x,y,t) = x^4\cosh t + y^4\sinh t, \tag{48}$$

in closed form.

Example 4.6. Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the wave-like models (Noor and Mohyud-Din (2008) and Wazwaz and Gorguis (2004)

$$U_{tt} = (x^2+y^2+z^2) + \frac{1}{2}(x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}),\;\; 0<x,\; y,\; z<1,\;\; t>0, \tag{49}$$

subject to the boundary conditions

$$\begin{aligned} &U(0,y,z,t)=y^2(e^t-1)+z^2(e^{-t}-1), && U(1,y,z,t)=(1+y^2)(e^t-1)+z^2(e^{-t}-1),\\ &U(x,0,z,t)=x^2(e^t-1)+z^2(e^{-t}-1), && U(x,1,z,t)=(1+x^2)(e^t-1)+z^2(e^{-t}-1),\\ &U(x,y,0,t)=(x^2+y^2)(e^t-1), && U(x,y,1,t)=(x^2+y^2)(e^t-1)+(e^{-t}-1), \end{aligned} \tag{50}$$

and having the initial conditions

$$U(x,y,z,0)=0,\ \ U_t(x,y,z,0)=x^2+y^2-z^2. \tag{51}$$

In a similar way as above, we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,y,z,t) &= (x^2+y^2+z^2)\frac{t^2}{2}\\ &+(x^2+y^2-z^2)t+p\left(\frac{1}{2}x^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t)\right)_{xx}\right]\right]\right.\\ &+\frac{1}{2}y^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t)\right)_{yy}\right]\right]\\ &\left.+\frac{1}{2}z^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t)\right)_{zz}\right]\right]\right). \end{aligned} \tag{52}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} &p^0: U_0(x,\mathrm{y},z,t)=\left(x^2+y^2+z^2\right)\frac{t^2}{2}+\left(x^2+y^2-z^2\right)t,\\ &p^1: U_1(x,\mathrm{y},z,t)=\left(x^2+y^2+z^2\right)\frac{t^4}{4!}+\left(x^2+y^2-z^2\right)\frac{t^3}{3!},\\ &p^2: U_2(x,\mathrm{y},z,t)=\left(x^2+y^2+z^2\right)\frac{t^6}{6!}+\left(x^2+y^2-z^2\right)\frac{t^5}{5!},\\ &p^3: U_3(x,\mathrm{y},z,t)=\left(x^2+y^2+z^2\right)\frac{t^8}{8!}+\left(x^2+y^2-z^2\right)\frac{t^7}{7!},\\ &\vdots \end{aligned} \tag{53}$$

Therefore the solution $U(x,\mathrm{y},z,t)$ is given by

$$U(x,\mathrm{y},z,t)=(x^2+y^2)\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)$$
$$+z^2\left(-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right), \qquad (54)$$

in the series form, and

$$U(x,\mathrm{y},z,t)=\left(x^2+y^2\right)e^t+z^2e^{-t}-\left(x^2+y^2+z^2\right), \qquad (55)$$

in closed form.

5. CONCLUSION

In this paper, we have applied the homotopy perturbation sumudu transform method (HPSTM) for solving heat and wave-like equations. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. The method gives more realistic series solutions that converge very rapidly in physical problems. The fact that the HPSTM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method. In conclusion, the HPSTM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

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